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Gabor-like systems in $\mathcal{L}^2(\mathbb{R}^d)$ and extensions to wavelets

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Abstract
In this paper we show how to construct a certain class of orthonormal bases in $\mathcal{L}^2(\mathbb{R}^d)$ starting from one or more Gabor orthonormal bases in $\mathcal{L}^2(\mathbb{R})$. Each such basis can be obtained acting on a single function $\Psi(\underline{x}) \in \mathcal{L}^2(\mathbb{R}^d)$ with a set of unitary operators which operate as translation and modulation operators *in suitable variables*. The same procedure is also extended to frames and wavelets. Many examples are discussed.

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1. Introduction

Gabor and wavelet systems of square-integrable functions are maybe the most interesting examples of (sometimes) orthonormal (o.n.) and complete systems in $\mathcal{L}^2(\mathbb{R})$. This is both for purely mathematical reasons but also in view of their applications to many different fields of physics, see for instance [9, 13]. This has produced a deep interest in these subjects and many mathematicians (but not only) have been involved in their analysis. In particular, one of the hardest problems to be considered was the construction of these systems and the analysis of their properties. This problem has been solved for wavelets by Mallat [10], introducing the so-called multi-resolution analysis. Since then, many other results have been found and many examples of o.n. wavelet bases have been constructed. Gabor systems have received much attention as well and their analysis has produced results which are often related to coherent states and frames. Necessary conditions for a Gabor frame to be an o.n. basis in $\mathcal{L}^2(\mathbb{R})$ can be found, for instance, in [8] and references therein. A detailed analysis of Gabor frames can be found in [6].

In two recent papers [1, 4], the authors discussed the possibility of getting an o.n. Gabor basis in $\mathcal{L}^2(\mathbb{R})$ (or in some closed subspace) starting from a non-orthogonal Gabor system. This was shown to be possible under suitable *density* conditions and a rather natural perturbative scheme has been proposed.

Here we continue our analysis of Gabor systems and show how a non-trivial Gabor-like o.n. basis in $\mathcal{L}^2(\mathbb{R}^d)$ can be constructed starting from one or more o.n. Gabor bases in $\mathcal{L}^2(\mathbb{R})$. We call this basis *Gabor-like* since it can be still obtained acting on a single function $\Psi(\underline{x}) \in \mathcal{L}^2(\mathbb{R}^d)$ with a set on unitary operators which behave like modulation and translation operators but only *in suitable variables*. We will clarify this point in the following. We also extend these results to general frames and to wavelets. In more detail, the paper is organized as follows.

In the next section we introduce the (physical and) mathematical framework which is used in the rest of the paper.

In section 3 we show how to construct an o.n. Gabor-like o.n. basis in two dimensions starting from two Gabor one-dimensional o.n. bases, which can coincide or not. We further extend this procedure to arbitrary dimensions and to frames.

Section 4 is devoted to some examples of our construction.

In section 5 we adapt the same strategy to produce o.n. wavelet-like bases in $\mathcal{L}^2(\mathbb{R}^d)$ starting from d o.n. wavelet bases in $\mathcal{L}^2(\mathbb{R})$. As before, we call these new sets *wavelets-like bases* since they are constructed starting from a single square integrable function $\Psi(\underline{x})$ acting on this with a set on unitary operators which behave like dilation and translation operators *in suitable variables*, as before. We end the section with some examples of our construction.

Section 6 contains our conclusions and projects for the future.

2. The mathematical setting

This section is devoted to the construction of the mathematical setting which is going to be used in our analysis. In what follows we will only consider a two-dimensional situation. The details of the extension to higher dimensionality are trivial and are left to the reader.

Consider the operators $\left(\left(\hat{x}_1^{(o)}, \hat{p}_1^{(o)}\right), \left(\hat{x}_2^{(o)}, \hat{p}_2^{(o)}\right)\right)$ and $\left(\left(\hat{x}_1^{(n)}, \hat{p}_1^{(n)}\right), \left(\hat{x}_2^{(n)}, \hat{p}_2^{(n)}\right)\right)$, acting on a certain Hilbert space $\mathcal{H}$ and satisfying

$$\left[\hat{x}_j^{(\alpha)}, \hat{p}_k^{(\alpha)}\right] = \mathrm{i}\delta_{jk}, \tag{2.1}$$

where $j, k = 1, 2$ and $\alpha = o, n$. Here o and n stand for *old* and *new*, since what we are interested in is to consider a unitary transformation from the canonical operators $\left(\hat{x}_j^{(o)}, \hat{p}_j^{(o)}\right)$, the *old* operators, into the new ones, $\left(\hat{x}_j^{(n)}, \hat{p}_j^{(n)}\right)$, $j = 1, 2$. We can now introduce the *generalized eigenvectors* of all these position operators:

$$\hat{x}_j^{(\alpha)} \xi_{x,[j]}^{(\alpha)} = x \xi_{x,[j]}^{(\alpha)}, \tag{2.2}$$

$j = 1, 2$, $\alpha = o, n$. Then we put, calling $\underline{x} = (x_1, x_2)$,

$$\xi_{\underline{x}}^{(\alpha)} = \xi_{x_1,[1]}^{(\alpha)} \otimes \xi_{x_2,[2]}^{(\alpha)}, \tag{2.3}$$

where, as before, $\alpha = o, n$. These vectors satisfy the following well-known properties [11]:

$$\left\langle \xi_{\underline{x}}^{(\alpha)}, \xi_{\underline{y}}^{(\alpha)} \right\rangle_{\mathcal{H}} = \delta(\underline{x} - \underline{y}) := \delta(x_1 - y_1)\delta(x_2 - y_2) \tag{2.4}$$

and

$$\int_{\mathbb{R}^2} \mathrm{d}\underline{x} \left| \xi_{\underline{x}}^{(\alpha)} \right\rangle \left\langle \xi_{\underline{x}}^{(\alpha)} \right| = \mathbf{1}_{\mathcal{H}}, \tag{2.5}$$

where we are adopting the Dirac bra–ket notation. Here $\alpha = o, n$ and we have introduced the subscript $\mathcal{H}$ since in the following we will need to distinguish between different scalar products in different Hilbert spaces. Any element $\Psi \in \mathcal{H}$ can be represented using the *new* or the *old* eigenvectors:

$$\Psi^{(\alpha)}(\underline{x}) := \left\langle \xi_{\underline{x}}^{(\alpha)}, \Psi \right\rangle_{\mathcal{H}}, \qquad \alpha = o, n. \tag{2.6}$$

This is a well-known procedure in quantum mechanics which corresponds to the possibility of using different representations (like the position and the momentum representations) to describe the same physical vector. It is quite easy to prove using (2.5) that, even if the generalized eigenvectors introduced in (2.2) and (2.3) do not belong to $\mathcal{H}$, both $\Psi^{(o)}(\underline{x})$ and $\Psi^{(n)}(\underline{x})$ belong to $\mathcal{L}^2(\mathbb{R}^2)$ and that $\|\Psi^{(n)}(\underline{x})\|_{\mathcal{L}^2(\mathbb{R}^2)} = \|\Psi^{(o)}(\underline{x})\|_{\mathcal{L}^2(\mathbb{R}^2)} = \|\Psi\|_{\mathcal{H}}$ for all $\Psi \in \mathcal{H}$. The two functions $\Psi^{(n)}(\underline{x})$ and $\Psi^{(o)}(\underline{x})$ are related to each other via the following equations:

$$\Psi^{(o)}(\underline{x}) = \int \mathrm{d}\underline{y}\big\langle \xi_{\underline{x}}^{(o)}, \xi_{\underline{y}}^{(n)}\big\rangle_{\mathcal{H}} \Psi^{(n)}(\underline{y}) = \int \mathrm{d}\underline{y}\, K(\underline{x};\underline{y})\Psi^{(n)}(\underline{y}) \tag{2.7}$$

and

$$\Psi^{(n)}(\underline{x}) = \int \mathrm{d}\underline{y}\big\langle \xi_{\underline{x}}^{(n)}, \xi_{\underline{y}}^{(o)}\big\rangle_{\mathcal{H}} \Psi^{(o)}(\underline{y}) = \int \mathrm{d}\underline{y}\, \overline{K(\underline{y};\underline{x})}\Psi^{(o)}(\underline{y}), \tag{2.8}$$

where we have introduced the *kernel* of the transformation,

$$K(\underline{x};\underline{y}) = \big\langle \xi_{\underline{x}}^{(o)}, \xi_{\underline{y}}^{(n)}\big\rangle_{\mathcal{H}}. \tag{2.9}$$

Since the operators $\hat{x}_j^{(\alpha)}$ and $\hat{p}_k^{(\beta)}$ are unbounded, they act on a dense domain $\mathcal{D}_{\mathcal{H}}$ of $\mathcal{H}$. Moreover, the action of, say, $\hat{x}_j^{(\alpha)}$ on the set $\mathcal{D}_{\text{gen}}$ of the generalized eigenvectors introduced in (2.2) and (2.3) is also well defined, see (2.2) and [11]. For this reason the matrix element $\big\langle f, \hat{x}_j^{(\alpha)} g\big\rangle_{\mathcal{H}}$, which makes no sense for general $f, g \in \mathcal{H}$, is well defined if, for instance, g belongs to $\mathcal{D}_{\text{gen}}$ or to $\mathcal{D}_{\mathcal{H}}$ and $f \in \mathcal{H}$. In the first case, i.e. if we consider $\big\langle f, \hat{x}_j^{(\alpha)}\xi_{\underline{x}}^{(\alpha)}\big\rangle_{\mathcal{H}}$, the result of this computation is a function of $\underline{x}$, $x_j\big\langle f, \xi_{\underline{x}}^{(\alpha)}\big\rangle_{\mathcal{H}} = x_j\overline{f^{(\alpha)}(\underline{x})}$, which is not necessarily square integrable (while $f^{(\alpha)}(\underline{x}) \in \mathcal{L}^2(\mathbb{R}^2)$). We can now introduce other operators, $\hat{X}_j^{(\alpha)}$ and $\hat{P}_k^{(\alpha)}$, which we often call in the rest of the paper the *capital operators*, via the equations

$$\hat{X}_j^{(\alpha)}\langle f, g\rangle_{\mathcal{H}} := \big\langle f, \hat{x}_j^{(\alpha)} g\big\rangle_{\mathcal{H}}, \qquad \hat{P}_j^{(\alpha)}\langle f, g\rangle_{\mathcal{H}} := \big\langle f, \hat{p}_j^{(\alpha)} g\big\rangle_{\mathcal{H}}, \tag{2.10}$$

for all α and j, where f and g are as above, and in particular they are such that the right-hand sides in (2.10) are well defined. For instance we have $\hat{X}_j^{(\alpha)}\big\langle f, \xi_{\underline{x}}^{(\alpha)}\big\rangle_{\mathcal{H}} = \big\langle f, \hat{x}_j^{(\alpha)}\xi_{\underline{x}}^{(\alpha)}\big\rangle_{\mathcal{H}} = x_j\big\langle f, \xi_{\underline{x}}^{(\alpha)}\big\rangle_{\mathcal{H}}$. These new operators are still canonically conjugate since they satisfy $\big[\hat{X}_j^{(\alpha)}, \hat{P}_k^{(\alpha)}\big] = \mathrm{i}\delta_{jk}$ (on a suitable domain).

The definition above produces a particularly relevant expression if we introduce the following unitary operators:

$$t_+(\underline{a}) = \mathrm{e}^{\mathrm{i}\underline{\hat{x}}^{(n)}\cdot\underline{a}}, \qquad t_-(\underline{b}) = \mathrm{e}^{\mathrm{i}\underline{\hat{p}}^{(n)}\cdot\underline{b}} \tag{2.11}$$

and their capital counterparts $T_+(\underline{a}) = \mathrm{e}^{\mathrm{i}\underline{\hat{X}}^{(n)}\cdot\underline{a}}$ and $T_-(\underline{b}) = \mathrm{e}^{\mathrm{i}\underline{\hat{P}}^{(n)}\cdot\underline{b}}$. Here $\underline{a} = (a_1, a_2)$, $\underline{b} = (b_1, b_2)$, $\underline{\hat{X}}^{(n)} = \big(\hat{X}_1^{(n)}, \hat{X}_2^{(n)}\big)$, $\underline{\hat{P}}^{(n)} = \big(\hat{P}_1^{(n)}, \hat{P}_2^{(n)}\big)$, $\underline{\hat{x}}^{(n)}\cdot\underline{a} = \hat{x}_1^{(n)}a_1 + \hat{x}_2^{(n)}a_2$, and so on. In this case we have, for instance,

$$T_+(\underline{a})f^{(\alpha)}(\underline{x}) = T_+(\underline{a})\big\langle \xi_{\underline{x}}^{(\alpha)}, f\big\rangle_{\mathcal{H}} = \big\langle \xi_{\underline{x}}^{(\alpha)}, t_+(\underline{a})f\big\rangle_{\mathcal{H}} \tag{2.12}$$

and

$$T_-(\underline{b})f^{(\alpha)}(\underline{x}) = T_-(\underline{b})\big\langle \xi_{\underline{x}}^{(\alpha)}, f\big\rangle_{\mathcal{H}} = \big\langle \xi_{\underline{x}}^{(\alpha)}, t_-(\underline{b})f\big\rangle_{\mathcal{H}}, \tag{2.13}$$

for all $f \in \mathcal{H}, \alpha = o, n$.

As it is clear the operators $T_+(\underline{a})$ and $T_-(\underline{b})$ are modulation and translation acting on $\mathcal{L}^2(\mathbb{R}^2)$ (but in the new variables, see below). Moreover they satisfy

$$T_+(\underline{a})T_-(\underline{b}) = \mathrm{e}^{-\mathrm{i}\underline{a}\cdot\underline{b}}T_-(\underline{b})T_+(\underline{a}), \tag{2.14}$$

and an analogous equality holds for $t_+(\underline{a})$ and $t_-(\underline{b})$.

Remark. It is worth noting that we are now introducing an asymmetry between the *old* and the *new* operators since, for reasons that will appear clear in the next section, we just need to define those unitary operators associated with $\left(\hat{x}_j^{(n)}, \hat{p}_j^{(n)}\right)$ and $\left(\hat{X}_j^{(n)}, \hat{P}_j^{(n)}\right)$, and not with the old ones.

3. Gabor-like o.n. bases in any dimension

The question we now want to address is quite a natural one and looks as follows: calling $\underline{a}(\underline{l}) = (a_1 l_1, a_2 l_2)$ and $\underline{b}(\underline{k}) = (b_1 k_1, b_2 k_2)$ we ask whether it is possible to find a square-integrable function, which we call $\Psi^{(o)}(\underline{x}) \in \mathcal{L}^2(\mathbb{R}^2)$ for reasons that will appear clear in a moment, such that the set

$$\mathcal{S} = \left\{\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x}) = T_+(\underline{a}(\underline{l}))T_-(\underline{b}(\underline{k}))\Psi^{(o)}(\underline{x}), \underline{k}, \underline{l} \in \mathbb{Z}^2\right\} \tag{3.1}$$

is an o.n. basis in $\mathcal{L}^2(\mathbb{R}^2)$.

We would like to stress that the problem is only close but not identical to the construction of a Gabor o.n. basis in $\mathcal{L}^2(\mathbb{R}^2)$. This is because, as we have already remarked before, the $T_\pm$ operators depend on the **new** operators while $\Psi^{(o)}(\underline{x}) = \left\langle \xi^{(o)}_{\underline{x}}, \Psi\right\rangle_{\mathcal{H}}$ is the projection of Ψ on the generalized eigenvectors of the **old** position operators. For this reason the set $\mathcal{S}$ is not obtained from $\Psi^{(o)}(\underline{x})$ via *canonical* translations and modulations and this is why we call $\mathcal{S}$ a *Gabor-like* rather than simply a *Gabor* set. This will be made explicit in the next section, when many examples will be discussed.

As already mentioned this problem is quite close to that addressed in [1, 4], where the authors have discussed a general strategy to construct, starting from a non o.n. set $\left\{f_{k_1,\ldots,k_N} := A_1^{k_1}\cdots A_N^{k_N} f_0, k_1,\ldots,k_N \in \mathbb{Z}\right\}$ where $f_0 \in \mathcal{H}$ is a fixed element and $A_1,\ldots,A_N$ are N given unitary operators, a different o.n. set, again of the same form $\left\{A_1^{k_1}\cdots A_N^{k_N} h_0, k_1,\ldots,k_N \in \mathbb{Z}\right\}$. For that we have discussed how to get such a vector $h_0 \in \mathcal{H}$ and we have shown that the procedure of orthogonalization usually fails to work in all of $\mathcal{H}$ while it works perfectly in suitable closed subspaces of $\mathcal{H}$, where the perturbation expansions adopted in [1, 4] converge quite fast. In the present settings the unitary operators are T_+ and T_- and our unknown is the function $\Psi^{(o)}(\underline{x})$. The procedure we are going to develop here is rather general, non-perturbative, and produces o.n. bases for all of $\mathcal{L}^2(\mathbb{R}^2)$.

Let us now go back to our original question. First we rewrite $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})$ in a convenient form. This can be done as follows:

$$\begin{aligned}\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x}) &= T_+(\underline{a}(\underline{l}))T_-(\underline{b}(\underline{k}))\Psi^{(o)}(\underline{x}) = \left\langle \xi^{(o)}_{\underline{x}}, t_+(\underline{a}(\underline{l}))t_-(\underline{b}(\underline{k}))\Psi\right\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, \left\langle \xi^{(o)}_{\underline{x}}, \xi^{(n)}_{\underline{y}}\right\rangle_{\mathcal{H}} \left\langle \xi^{(n)}_{\underline{y}}, t_+(\underline{a}(\underline{l}))t_-(\underline{b}(\underline{k}))\Psi\right\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, K(\underline{x};\underline{y})\, \mathrm{e}^{\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\Psi^{(n)}(\underline{y}+\underline{b}(\underline{k}))\end{aligned} \tag{3.2}$$

where we have used (2.9), (2.12), (2.13), the resolution of the identity for $\left\{\xi^{(n)}_{\underline{y}}\right\}$, and the fact that, as it easily checked, $t_-(\underline{b}(\underline{k}))^\dagger t_+(\underline{a}(\underline{l}))^\dagger \xi^{(n)}_{\underline{y}} = \mathrm{e}^{-\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\xi^{(n)}_{(y_1+b_1k_1, y_2+b_2k_2)} = \mathrm{e}^{-\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\xi^{(n)}_{\underline{y}+\underline{b}(\underline{k})}$.

The next step consists now in computing the following scalar product in $\mathcal{L}^2(\mathbb{R}^2)$:

$$\Omega_{\underline{l},\underline{k}} := \left\langle \Psi^{(o)}_{\underline{l},\underline{k}}, \Psi^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \mathrm{d}\underline{x}\, \overline{\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})}\Psi^{(o)}(\underline{x}).$$

Using formula (3.2) and the resolution of the identity for the set $\left\{\xi^{(o)}_{\underline{x}}\right\}$ we deduce that

$$\Omega_{\underline{l},\underline{k}} = \int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, \mathrm{e}^{-\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\overline{\Psi^{(n)}(\underline{y}+\underline{b}(\underline{k}))}\Psi^{(n)}(\underline{y}). \tag{3.3}$$

This last integral can be written in a factorized form if $\Psi^{(n)}(\underline{y})$ is, by itself, factorizable. This does not necessarily imply that $\Psi^{(o)}(\underline{x})$ can also be written as the product of a function which only depends on x_1 times another function which depends on x_2. We will come back to this point in the next section where our examples will explicitly show this claim. This implies, of course, that our approach is rather different (and more interesting) than simply taking tensor products. Let us therefore assume that

$$\Psi^{(n)}(\underline{y}) = h_1(y_1)h_2(y_2). \tag{3.4}$$

Hence we find

$$\Omega_{\underline{l},\underline{k}} = \left(\int_{\mathbb{R}} \mathrm{d}y_1\, \mathrm{e}^{-\mathrm{i}y_1a_1l_1}\overline{h_1(y_1+b_1k_1)}h_1(y_1)\right)\left(\int_{\mathbb{R}} \mathrm{d}y_2\, \mathrm{e}^{-\mathrm{i}y_2a_2l_2}\overline{h_2(y_2+b_2k_2)}h_2(y_2)\right). \tag{3.5}$$

This is the first ingredient for the proof of proposition 1 below. Before stating the main result of this section, however, we still need to compute the scalar product (in $\mathcal{L}^2(\mathbb{R}^2)$) between a generic element $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})$ of the set $\mathcal{S}$ and a function $f^{(o)}(\underline{x})$ of $\mathcal{L}^2(\mathbb{R}^2)$. This computation is not very different from that in (3.3). Indeed, the same steps can be repeated and we find that

$$\left\langle\Psi^{(o)}_{\underline{l},\underline{k}}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \mathrm{d}\underline{x}\, \overline{\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})}\, f^{(o)}(\underline{x}) = \int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, \mathrm{e}^{-\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\,\overline{\Psi^{(n)}(\underline{y}+\underline{b}(\underline{k}))}\, f^{(n)}(\underline{y}) \tag{3.6}$$

which gives the scalar product of the two functions in *the new representation*. It is further convenient to write this equation in a different way, by introducing the function

$$F_{l_2,k_2}(y_1) := \int_{\mathbb{R}} \mathrm{d}y_2\, \mathrm{e}^{-\mathrm{i}y_2a_2l_2}\overline{h_2(y_2+b_2k_2)}\, f^{(n)}(y_1,y_2). \tag{3.7}$$

We get

$$\left\langle\Psi^{(o)}_{\underline{l},\underline{k}}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}} \mathrm{d}y_1\, \mathrm{e}^{-\mathrm{i}y_1a_1l_1}\overline{h_1(y_1+b_1k_1)}F_{l_2,k_2}(y_1). \tag{3.8}$$

It is evident from the previous formulae that a relevant role in all our computations is played by the one-dimensional Gabor systems generated by h_1 and h_2: $s_j = \left\{h^{(j)}_{l,k}(x) := \mathrm{e}^{\mathrm{i}xa_jl}h_j(x+b_jk), k,l \in \mathbb{Z}\right\}$, $j = 1, 2$.

The following proposition holds true:

Proposition 1. *Let s_1 and s_2 be two o.n. bases in $\mathcal{L}^2(\mathbb{R})$. Then the set $\mathcal{S}$ is an o.n. basis in $\mathcal{L}^2(\mathbb{R}^2)$.*

Proof. We first prove that the set $\mathcal{S}$ is orthonormal. For that we start remarking that our assumptions on s_1 and s_2, together with (3.5), imply that $\Omega_{\underline{l},\underline{k}} = \left\langle\Psi^{(o)}_{\underline{l},\underline{k}}, \Psi^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \delta_{\underline{l},\underline{0}}\delta_{\underline{k},\underline{0}}$. This result, together with easy properties of the operators $T_\pm$, implies in turn $\left\langle\Psi^{(o)}_{\underline{l},\underline{k}}, \Psi^{(o)}_{\underline{n},\underline{m}}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \delta_{\underline{l},\underline{n}}\delta_{\underline{k},\underline{m}}$.

To prove that $\mathcal{S}$ is complete in $\mathcal{L}^2(\mathbb{R}^2)$ we assume that $\left\langle\Psi^{(o)}_{\underline{l},\underline{k}}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = 0$ for all $\underline{l}, \underline{k} \in \mathbb{Z}^2$. Hence, because of (3.8) and of the completeness of the set s_1, $F_{l_2,k_2}(y_1) = 0$ almost everywhere (a.e.) in $\mathbb{R}$ for all possible choices of l_2 and k_2 in $\mathbb{Z}$. As a consequence of this, of (3.7) and of the completeness of s_2, we conclude that the function $g_{y_1}(y_2) := f^{(n)}(y_1, y_2)$ is also zero a.e., which finally implies that $f^{(o)}(\underline{x}) = \int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, K(\underline{x};\underline{y})f^{(n)}(\underline{y})$ is zero a.e. as well. □

Remark. It might be worth remarking once again that the procedure is only apparently trivial since, even if $\Psi^{(n)}(\underline{y})$ is factorized, $\Psi^{(o)}(\underline{x})$ needs not to be. We will show this in the examples discussed in the next section.

3.1. Extensions to $d > 2$ and to frames

It is not very hard to imagine that this procedure can be extended to any dimension. In this case we start with a set of d *old* conjugate operators $\left(\hat{x}_j^{(o)}, \hat{p}_j^{(o)}\right)$, $j = 1, 2, \ldots, d$, which are mapped, via a canonical transformation, into a *new* set of d conjugate operators $\left(\hat{x}_j^{(n)}, \hat{p}_j^{(n)}\right)$, $j = 1, 2, \ldots, d$. As before we can introduce the generalized eigenvectors of all these position operators, their tensor product $\xi_{\underline{x}}^{(\alpha)} = \xi_{x_1,[1]}^{(\alpha)} \otimes \xi_{x_2,[2]}^{(\alpha)} \otimes \cdots \otimes \xi_{x_d,[d]}^{(\alpha)}$, $\alpha = o, n$, and these vectors satisfy a δ-like normalization and a resolution of the identity, see (2.4) and (2.5). Then, if we put $\Psi^{(\alpha)}(\underline{x}) := \left\langle \xi_{\underline{x}}^{(\alpha)}, \Psi\right\rangle_{\mathcal{H}}$, we can change between *old* and *new* variables and the relations are given in (2.7)–(2.9). Moreover, we can introduce *capital* and *small* unitary operators like in (2.11)–(2.13). For instance we have $t_+(\underline{a}) = \mathrm{e}^{\mathrm{i}\underline{\hat{x}}^{(n)}\cdot\underline{a}}$, where $\underline{\hat{x}}^{(n)} \cdot \underline{a} = \hat{x}_1^{(n)} a_1 + \cdots + \hat{x}_d^{(n)} a_d$.

As before the question is the following: is it possible to find a function $\Psi^{(o)}(\underline{x}) \in \mathcal{L}^2(\mathbb{R}^d)$ such that the set $\mathcal{S} = \left\{\Psi_{\underline{l},\underline{k}}^{(o)}(\underline{x}) = T_+(\underline{a}(\underline{l}))T_-(\underline{b}(\underline{k}))\Psi^{(o)}(\underline{x}), \underline{k}, \underline{l} \in \mathbb{Z}^2\right\}$ is an o.n. basis in $\mathcal{L}^2(\mathbb{R}^d)$? Here, extending our previous notation, we have $\underline{a}(\underline{l}) = (a_1 l_1, \ldots, a_d l_d)$ and $\underline{b}(\underline{k}) = (b_1 k_1, \ldots, b_d k_d)$. Once again the answer is positive and the procedure is an almost trivial extension of that discussed above. We deduce that

$$\Psi_{\underline{l},\underline{k}}^{(o)}(\underline{x}) = \int_{\mathbb{R}^d} \mathrm{d}\underline{y}\, K(\underline{x}; \underline{y})\, \mathrm{e}^{\mathrm{i}\underline{y}\cdot\underline{a}(\underline{l})}\, \Psi^{(n)}(\underline{y} + \underline{b}(\underline{k})),$$

where $K(\underline{x}; \underline{y}) = \left\langle \xi_{\underline{x}}^{(o)}, \xi_{\underline{y}}^{(n)}\right\rangle_{\mathcal{H}}$, and, if $\Psi^{(n)}(\underline{y}) = h_1(y_1)\cdots h_d(y_d)$, then

$$\Omega_{\underline{l},\underline{k}} := \left\langle \Psi_{\underline{l},\underline{k}}^{(o)}, \Psi^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^d)} = \prod_{j=1}^{d} \left(\int_{\mathbb{R}} \mathrm{d}y_j\, \mathrm{e}^{-\mathrm{i}y_j a_j l_j}\, \overline{h_j(y_j + b_j k_j)}\, h_j(y_j) \right),$$

while formula (3.8) must be replaced by

$$\left\langle \Psi_{\underline{l},\underline{k}}^{(o)}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}} \mathrm{d}y_1\, \mathrm{e}^{-\mathrm{i}y_1 a_1 l_1}\, \overline{h_1(y_1 + b_1 k_1)}\, F_{l_2,k_2,\cdots,l_d,k_d}^{(1)}(y_1) \tag{3.9}$$

where

$$\begin{aligned} F_{l_2,k_2,\cdots,l_d,k_d}^{(1)}(y_1) := \int_{\mathbb{R}^{d-1}} \mathrm{d}y_2 \ldots \mathrm{d}y_d\, \mathrm{e}^{-\mathrm{i}(y_2 a_2 l_2 + \cdots + y_d a_d l_d)} \\ \times \overline{h_2(y_2 + b_2 k_2)} \cdots \overline{h_d(y_d + b_d k_d)}\, f^{(n)}(y_1, y_2, \ldots, y_d). \end{aligned} \tag{3.10}$$

Needless to say, in this case we need to introduce d different one-dimensional Gabor systems $s_j = \{\mathrm{e}^{\mathrm{i}x a_j l} h_j(x + b_j k), k, l \in \mathbb{Z}\}$, $j = 1, 2, \ldots, d$, and if each one of these is an o.n. basis in $\mathcal{L}^2(\mathbb{R})$,[1] then it is trivial to check that $\mathcal{S}$ is an o.n. set in $\mathcal{L}^2(\mathbb{R}^d)$. As for its completeness, our previous proof can also be extended in this case: from (3.9) and from the completeness of s_1 we first deduce that $F_{l_2,k_2,\cdots,l_d,k_d}^{(1)}(y_1)$ is zero a.e. for all possible integer choices of $l_2, k_2, \ldots, l_d$ and k_d. But, because of (3.10) and of the completeness of s_2, also the function

$$\begin{aligned} F_{l_3,k_3,\cdots,l_d,k_d}^{(2)}(y_1, y_2) := \int_{\mathbb{R}^{d-2}} \mathrm{d}y_3 \ldots \mathrm{d}y_d\, \mathrm{e}^{-\mathrm{i}(y_3 a_3 l_3 + \cdots + y_d a_d l_d)} \\ \times \overline{h_3(y_3 + b_3 k_3)} \cdots \overline{h_d(y_d + b_d k_d)}\, f^{(n)}(y_1, y_2, \ldots, y_d) \end{aligned}$$

is zero a.e. for all possible $l_3, k_3, \ldots, l_d, k_d \in \mathbb{Z}$. Iterating this procedure we finally conclude that $f^{(n)}(\underline{y}) = 0$ a.e. and therefore $f^{(o)}(\underline{x}) = 0$.

We postpone to the next section one example of this construction. Now we want to show how this same procedure can also be adapted to construct frames in d dimensions starting from

[1] Of course they could all be coincident. In this case it is enough to start with a single o.n. Gabor basis in $\mathcal{L}^2(\mathbb{R})$.

d frames in one dimension. For the reader's convenience we recall that an (A, B)-frame of an Hilbert space $\mathcal{H}$, $0 < A \leqslant B < \infty$, is a set of vectors $\{\varphi_n \in \mathcal{H}, n \in J\}$, labeled by a set $J \subseteq \mathbb{N}$, such that the inequality $A\|f\|^2 \leqslant \sum_{n\in J} |\langle \varphi_n, f\rangle|^2 \leqslant B\|f\|^2$ holds for any $f \in \mathcal{H}$.

In particular we can prove the following:

Proposition 2. *Let s_j be an (A_j, B_j)-frame, $j = 1, 2, \ldots, d$, in $\mathcal{L}^2(\mathbb{R})$. Then the set $\mathcal{S}$ is an $(A_1A_2\cdots A_d, B_1B_2\cdots B_d)$-frame in $\mathcal{L}^2(\mathbb{R}^d)$.*

Proof. We give the proof of this statement only for $d = 2$. The extension to higher dimensions is straightforward.

Using (3.8) and the definition of s_1 we deduce that

$$\sum_{\underline{k},\underline{l}\in\mathbb{Z}^2} \left|\left\langle \Psi^{(o)}_{\underline{l},\underline{k}}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)}\right|^2 = \sum_{k_2,l_2\in\mathbb{Z}} \left(\sum_{k_1,l_1\in\mathbb{Z}} \left|\left\langle h^{(1)}_{l_1,k_1}, F_{l_2,k_2}\right\rangle_{\mathcal{L}^2(\mathbb{R})}\right|^2 \right),$$

so that, because of our assumption on s_1, we get

$$A_1 \sum_{k_2,l_2\in\mathbb{Z}} \|F_{l_2,k_2}\|^2_{\mathcal{L}^2(\mathbb{R})} \leqslant \sum_{\underline{k},\underline{l}\in\mathbb{Z}^2} \left|\left\langle \Psi^{(o)}_{\underline{l},\underline{k}}, f^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)}\right|^2 \leqslant B_1 \sum_{k_2,l_2\in\mathbb{Z}} \|F_{l_2,k_2}\|^2_{\mathcal{L}^2(\mathbb{R})}.$$

Now, since s_2 is an (A_2, B_2)-frame, because of (3.7) it is not hard to prove that

$$A_2\|f^{(n)}\|^2_{\mathcal{L}^2(\mathbb{R}^2)} \leqslant \sum_{k_2,l_2\in\mathbb{Z}} \|F_{l_2,k_2}\|^2_{\mathcal{L}^2(\mathbb{R})} \leqslant B_2\|f^{(n)}\|^2_{\mathcal{L}^2(\mathbb{R}^2)}$$

which, together with our previous estimate and using the equality $\|f^{(n)}\|_{\mathcal{L}^2(\mathbb{R}^2)} = \|f^{(o)}\|_{\mathcal{L}^2(\mathbb{R}^2)}$, implies the statement. □

4. Examples

This section is devoted to discussing in detail some examples, concerning the construction above of an o.n. Gabor-like basis and of Gabor-like frame of $\mathcal{L}^2(\mathbb{R}^d)$. As we have discussed in the previous section, and it is clarified by formula (2.7), $\Psi^{(o)}(\underline{x}) = \int \mathrm{d}\underline{y}\, K(\underline{x}; \underline{y})\Psi^{(n)}(\underline{y})$, a special role in our construction is played by the kernel of the canonical transformation between $\left(\hat{x}^{(o)}_j, \hat{p}^{(o)}_j\right)$ and $\left(\hat{x}^{(n)}_j, \hat{p}^{(n)}_j\right)$, $j = 1, \ldots, d$. Of course the analytic expression of $K(\underline{x}; \underline{y}) = \left\langle \xi^{(o)}_{\underline{x}}, \xi^{(n)}_{\underline{y}}\right\rangle_{\mathcal{H}}$ strongly depends on the transformation itself and it can be deduced using the results in [12]. Some of the kernels we are going to consider here are related to physically relevant transformations, already considered by the author in different contexts. As for the o.n. Gabor systems in $d = 1$, we will consider here the following two well-known sets, [8]:

$$\mathcal{G}^{(j)} = \left\{g^{(j)}_{m,n}(x) = \mathrm{e}^{2\pi \mathrm{i} m x} g^{(j)}(x - n), m, n \in \mathbb{Z}\right\}, \tag{4.1}$$

where $g^{(1)}(x) = \chi_{[0,1]}(x)$ or $g^{(2)}(x) = \frac{\sin(\pi x)}{\pi x}$. Here $\chi_{[0,1]}(x)$ is the characteristic function of the interval [0, 1].

4.1. Quantum Hall effect

In the analysis of the quantum Hall effect a particularly relevant operator is the Hamiltonian of a single electron subjected to a strong and uniform magnetic field along z. This Hamiltonian, in the so-called symmetric gauge, looks like

$$H = \frac{1}{2}(\underline{p} + \underline{A}(\underline{r}))^2 = \frac{1}{2}\left(p_x - \frac{y}{2}\right)^2 + \left(p_y + \frac{x}{2}\right)^2,$$

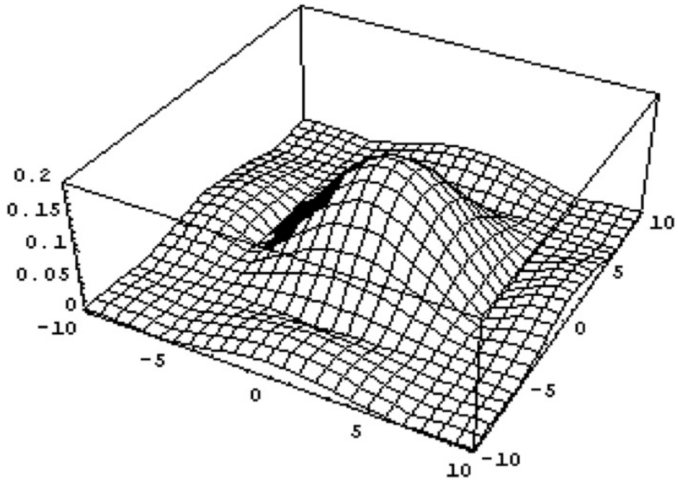

Figure 1. $|\Psi^{(o)}(\underline{x})|$ for $h_1(x) = h_2(x) = \chi_{[0,1]}(x)$.

which can be written, identifying x, y, p_x and p_y respectively with $\hat{x}_1^{(o)}, \hat{x}_2^{(o)}, \hat{p}_1^{(o)}$ and $\hat{p}_2^{(o)}$, as $H = \frac{1}{2}\left(\hat{p}_1^{(o)} - \hat{x}_2^{(o)}/2\right)^2 + \left(\hat{p}_2^{(o)} + \hat{x}_1^{(o)}/2\right)^2$. If we now introduce the following canonical transformation, see [2] and references therein,

$$\hat{x}_1^{(n)} = \hat{p}_1^{(o)} + \frac{\hat{x}_2^{(o)}}{2}, \quad \hat{x}_2^{(n)} = \hat{p}_2^{(o)} + \frac{\hat{x}_1^{(o)}}{2}, \quad \hat{p}_1^{(n)} = \hat{p}_2^{(o)} - \frac{\hat{x}_1^{(o)}}{2}, \quad \hat{p}_2^{(n)} = \hat{p}_1^{(o)} - \frac{\hat{x}_2^{(o)}}{2}, \tag{4.2}$$

H can be written as $H = \frac{1}{2}\left(\left(\hat{p}_2^{(n)}\right)^2 + \left(\hat{x}_2^{(n)}\right)^2\right)$. This, because of the canonicity of the transformation, is just the Hamiltonian of a quantum harmonic oscillator in the variables $\left(\hat{x}_2^{(n)}, \hat{p}_2^{(n)}\right)$. The physical reason for introducing (4.2) is that it displays rather clearly the infinite degeneracy of the so-called *Landau levels*. This means that, if $\varphi(x)$ is an eigenstate of H with eigenvalue E, then $\mathrm{e}^{\mathrm{i}\hat{x}_1^{(n)}\alpha}\,\mathrm{e}^{\mathrm{i}\hat{p}_1^{(n)}\beta}\varphi(x)$ is still an eigenstate of H, corresponding to the same eigenvalue, for all possible choices of α and β. This and many other aspects of the transformation in (4.2), together with its possible application to the theory of multi-resolution analysis, have been discussed in [3] and references therein. The kernel $K(\underline{x};\underline{y})$ can be found using the results in [12] and in [7]. We get

$$K(\underline{x};\underline{y}) = \frac{1}{2\pi}\exp\left\{\mathrm{i}\left(x_1y_1 + x_2y_2 - y_1y_2 - \frac{x_1x_2}{2}\right)\right\}. \tag{4.3}$$

We are now ready to find the function $\Psi^{(o)}(\underline{x})$ for different choices of the *one-dimensional ingredients*, see (4.1), and considering the formula $\Psi^{(o)}(\underline{x}) = \int \mathrm{d}\underline{y}\, K(\underline{x};\underline{y})\Psi^{(n)}(\underline{y})$, with $\Psi^{(n)}(\underline{y}) = h_1(y_1)h_2(y_2)$.

We start considering here the following choice: $h_1(x) = h_2(x) = \chi_{[0,1]}(x)$. This is a symmetric choice and produces the following function:

$$\Psi^{(o)}(\underline{x}) = \frac{\mathrm{e}^{\mathrm{i}x_1x_2/2}}{2\pi\mathrm{i}}\int_{x_2-1}^{x_2}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}x_1t}\,\frac{\mathrm{e}^{\mathrm{i}t}-1}{t},$$

which can be computed explicitly and gives a rather involved analytic expression for $\Psi^{(o)}(\underline{x})$ which *is not factorizable*. Rather than giving this expression, we just give in figure 1 the plot of the modulus of $\Psi^{(o)}(\underline{x})$:

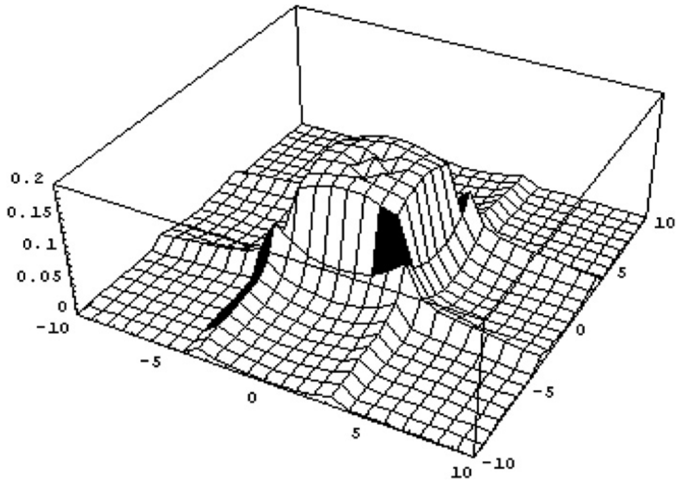

Figure 2. $|\Psi^{(o)}(\underline{x})|$ for $h_1(x) = h_2(x) = \frac{\sin(\pi x)}{\pi x}$.

As we can see this function is localized around the origin and goes to zero when $x^2 + y^2$ diverges.

Another example can be constructed choosing $h_1(x) = h_2(x) = \frac{\sin(\pi x)}{\pi x}$. In this case we get

$$\Psi^{(o)}(\underline{x}) = \frac{\mathrm{e}^{-\mathrm{i}x_1x_2/2}}{2\pi^2}\int_{x_2-\pi}^{x_2+\pi}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}x_1t}\frac{\sin(\pi t)}{t},$$

whose modulus is plotted in the figure 2.

We postpone other details of this construction to the next subsection, since in that situation the kernel will have a simpler form and this will simplify all our computations.

4.2. Another example from quantum mechanics

The example we discuss now arises from an old paper, [5], in which some aspects of the relevance of o.n. wavelet bases in the analysis of the quantum Hall effect were investigated. In particular we have constructed a two-dimensional quantum Hamiltonian, close to that of the quantum Hall effect, which again can be written in new variables as the Hamiltonian of a quantum oscillator. This was useful to show, via explicit computation of the Coulomb energy of a two-electrons system, that wavelet-like o.n. bases in the lowest Landau levels have very good localization features when compared to bases arising from Gaussian functions, [5]. In this case the canonical transformation is

$$\hat{x}_1^{(n)} = \hat{p}_1^{(o)} + \hat{p}_2^{(o)}, \qquad \hat{x}_2^{(n)} = \hat{p}_2^{(o)}, \qquad \hat{p}_1^{(n)} = -\hat{x}_1^{(o)}, \qquad \hat{p}_2^{(n)} = \hat{x}_1^{(o)} - \hat{x}_2^{(o)}, \tag{4.4}$$

and we find that, [5],

$$K(\underline{x};\underline{y}) = \frac{1}{2\pi}\exp\{\mathrm{i}\,(x_1y_1 + y_2(x_2 - x_1))\}. \tag{4.5}$$

If we compute again $\Psi^{(o)}(\underline{x}) = \int \mathrm{d}\underline{y}\,K(\underline{x};\underline{y})\Psi^{(n)}(\underline{y})$ for $\Psi^{(n)}(\underline{y}) = h_1(y_1)h_2(y_2)$ we have several different possibilities, which are considered in the following.

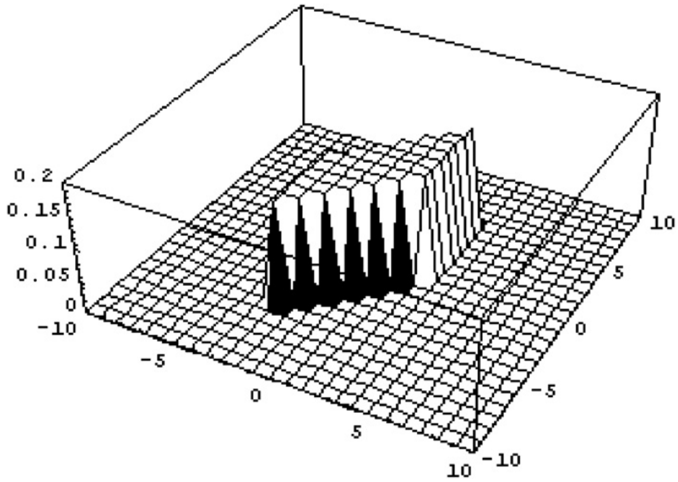

Figure 3. $|\Psi^{(o)}(\underline{x})|$ for $h_1(x) = h_2(x) = \frac{\sin(\pi x)}{\pi x}$.

Case 1: $h_1(x) = h_2(x) = \chi_{[0,1]}(x)$.

In this case the computation is easily performed and we get

$$\Psi^{(o)}(\underline{x}) = \frac{1}{2\pi}\frac{e^{ix_1}-1}{x_1}\frac{e^{i(x_2-x_1)}-1}{x_2-x_1},$$

which is clearly *not* the product of a function which depends on x_1 and another function which depends on x_2. This is another evidence of the fact that, as we have stated many times along the text, our approach is not merely a *tensor product technique* but really produces different functions which are genuinely in $\mathcal{L}^2(\mathbb{R}^2)$ and not necessarily in $\mathcal{L}^2(\mathbb{R})\otimes\mathcal{L}^2(\mathbb{R})$.

We also want to investigate, in the ambit of this example, the role of the operators $T_\pm$. We do it here because, in view of the simplicity of (4.4), it is easier to display the result. Let us recall that, because of (3.1), the entire set of functions in $\mathcal{L}^2(\mathbb{R}^2)$ looks like $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x}) = T_+(\underline{a}(\underline{l}))T_-(\underline{b}(\underline{k}))\Psi^{(o)}(\underline{x})$, where $T_+(\underline{a}(\underline{l})) = e^{i\hat{\underline{X}}^{(n)}\cdot\underline{a}(\underline{l})} = e^{i\hat{X}_1^{(n)}a_1l_1}\,e^{i\hat{X}_2^{(n)}a_2l_2}$ and $T_-(\underline{b}(\underline{k})) = e^{i\hat{\underline{P}}^{(n)}\cdot\underline{b}(\underline{k})} = e^{i\hat{P}_1^{(n)}b_1k_1}e^{i\hat{P}_2^{(n)}b_2k_2}$. Moreover, because of the transformation (4.4) (and of its counterpart for the capital operators), it is a standard computation to check that

$$\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x}) = e^{i(x_1+a_1l_1)(b_2k_2-b_1k_1)}\,e^{-i(x_2+\underline{a}\cdot\underline{l})b_2k_2}\,\Psi^{(o)}(\underline{x}+\underline{a}\cdot\underline{l}) \tag{4.6}$$

which is again the modulated and translated version of $\Psi^{(o)}(\underline{x})$, but in a slightly non-trivial way.

Case 2: $h_1(x) = h_2(x) = \frac{\sin(\pi x)}{\pi x}$.

This is again a symmetrical choice for which all the computations can be easily performed. The result can be written as

$$\Psi^{(o)}(\underline{x}) = \frac{1}{2\pi}\chi_{[-\pi,\pi]}(x_1)\chi_{[-\pi,\pi]}(x_2-x_1)$$

which is again not-factorizable. Figure 3 displays the plot of this function, which is clearly compactly supported. As for the functions $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})$, since they are related to the same canonical transformation, then they can be obtained as in case 1 and look as in (4.6).

Case 3: $h_1(x) = \frac{\sin(\pi x)}{\pi x}$ and $h_2(x) = \chi_{[0,1]}(x)$.

This choice is no longer symmetric and the result, which again can be easily computed, is the following:

$$\Psi^{(o)}(\underline{x}) = \frac{1}{2\pi}\chi_{[-\pi,\pi]}(x_1)\frac{e^{i(x_2-x_1)}-1}{x_2-x_1}.$$

Once again, the functions $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})$ can be computed as (4.6).

4.3. An example in $d = 3$

As we have widely discussed before the only real ingredient to apply our procedure is a canonical transformation from three *old* pairs of conjugate operators into three *new* pairs of similar operators. No physical motivation is strictly required, and indeed there is no motivation in the map we consider below, which is chosen only in view of its simplicity. We put

$$\begin{cases} \hat{x}_1^{(n)} = \frac{1}{2}\left(\hat{p}_3^{(o)} - \hat{p}_1^{(o)} - \hat{p}_2^{(o)}\right), & \hat{p}_1^{(n)} = \hat{x}_1^{(o)} + \hat{x}_2^{(o)} \\ \hat{x}_2^{(n)} = \frac{1}{2}\left(\hat{p}_1^{(o)} - \hat{p}_2^{(o)} - \hat{p}_3^{(o)}\right), & \hat{p}_2^{(n)} = \hat{x}_2^{(o)} + \hat{x}_3^{(o)} \\ \hat{x}_3^{(n)} = \frac{1}{2}\left(\hat{p}_2^{(o)} - \hat{p}_1^{(o)} - \hat{p}_3^{(o)}\right), & \hat{p}_3^{(n)} = \hat{x}_3^{(o)} + \hat{x}_1^{(o)}. \end{cases} \tag{4.7}$$

In this case the kernel is [12]

$$K(\underline{x};\underline{y}) = \frac{1}{2\pi^{3/2}}\exp\left\{-i\left(y_1(x_1+x_2) + y_2(x_2+x_3) + y_3(x_1+x_3)\right)\right\}. \tag{4.8}$$

If we now choose $\Psi^{(n)}(\underline{y}) = \chi_{[0,1]}(y_1)\chi_{[0,1]}(y_2)\chi_{[0,1]}(y_3)$ we get

$$\Psi^{(o)}(\underline{x}) = \frac{i}{2\pi^{3/2}}\frac{e^{-i(x_1+x_2)}-1}{x_1+x_2}\frac{e^{-i(x_2+x_3)}-1}{x_2+x_3}\frac{e^{-i(x_1+x_3)}-1}{x_1+x_3}.$$

Also this function is not the product of three one-dimensional functions, as claimed before. From this $\Psi^{(o)}(\underline{x})$ we can further deduce the form of the functions $\Psi^{(o)}_{\underline{l},\underline{k}}(\underline{x})$, following the same steps as in section 4.2. We leave to the reader the construction of other examples which can be obtained, for instance, considering different choices of the single functions defining $\Psi^{(n)}(\underline{y})$.

4.4. Two frames in $d = 2$

Our ingredient for this example is the following: $h(t) \equiv h_1(t) = h_2(t) = \frac{e^{-t^2/2}}{\pi^{1/4}}$, which is such that the set $e^{im\omega_o t}h(t - nt_o)$ is a frame for all possible choices $\omega_o t_o \leqslant 2\pi$.

In this case, if we compute $\Psi^{(o)}(\underline{x})$ using the kernel (4.3), we get

$$\Psi^{(o)}(\underline{x}) = \frac{1}{\sqrt{2\pi}}e^{-(x_1^2+x_2^2)/4}.$$

This result is not unexpected since it was well known for many years, see [2] and references therein, the role of this two-dimensional Gaussian in the analysis of the Landau levels arising in the analysis of the quantum Hall effect.

If, on the other hand, we choose the kernel in (4.5) we find, after few easy computations,

$$\Psi^{(o)}(\underline{x}) = \frac{1}{\sqrt{2\pi}}e^{-(x_1^2+(x_2-x_1)^2)/2}$$

which shows an asymmetric behavior between the variables x_1 and x_2.

5. What about wavelet?

The procedure discussed so far for Gabor-like systems can be extended to sets of wavelets. To deduce this extension it is convenient to give the first few mathematical results, complementing those discussed in section 2, and this will be done in the next subsection, while the explicit extension with two examples will be discussed in section 5.2.

5.1. Still more mathematics

In the case of wavelets the modulation operator in section 2 must be replaced by the dilation operator while the translation operator is still needed. As before, in our approach is convenient to pay attention to the difference between the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2(\mathbb{R}^2)$. In this case it maybe more natural to start defining the operators on $\mathcal{L}^2(\mathbb{R}^2)$ and then defining their *small counterparts* acting on $\mathcal{H}$. Note that this is exactly the opposite of what we have done in section 2. For this we begin defining the following operators, [6]:

$$\begin{cases} T_1 f^{(n)}(x_1, x_2) = f^{(n)}(x_1 - 1, x_2), \\ T_2 f^{(n)}(x_1, x_2) = f^{(n)}(x_1, x_2 - 1), \\ D_1 f^{(n)}(x_1, x_2) = \sqrt{2} f^{(n)}(2x_1, x_2), \\ D_2 f^{(n)}(x_1, x_2) = \sqrt{2} f^{(n)}(x_1, 2x_2), \end{cases} \qquad \forall f^{(n)}(x_1, x_2) = \left\langle \xi_{\underline{x}}^{(n)}, f \right\rangle_{\mathcal{H}} \in \mathcal{L}^2(\mathbb{R}^2). \tag{5.1}$$

Once again we only need to introduce operators which act as dilations and translations in the *new* variables. This is exactly the analogous of what we have done in section 2.

It is trivial to check that all these operators are unitary, that $[T_1, T_2] = [D_1, D_2] = [T_1, D_2] = [T_2, D_1] = 0$, and that

$$\begin{cases} T_m^k D_m^j = D_m^j T_m^{2^j k} \\ D_m^j T_m^k = T_m^{2^{-j} k} D_m^j \end{cases} \qquad m = 1, 2, \quad \forall j, k \in \mathbb{Z}. \tag{5.2}$$

If we now introduce the *small operators* t_m and d_m in the same spirit and repeating the same comments as in (2.10),

$$T_m \langle f, g \rangle_{\mathcal{H}} =: \langle f, t_m g \rangle_{\mathcal{H}}, \qquad D_m \langle f, g \rangle_{\mathcal{H}} := \langle f, d_m g \rangle_{\mathcal{H}}, \tag{5.3}$$

for $m = 1, 2$, it is easy to deduce that

$$\begin{cases} t_1 \xi_{\underline{x}}^{(n)} = \xi_{(x_1+1, x_2)}^{(n)}, & t_2 \xi_{\underline{x}}^{(n)} = \xi_{(x_1, x_2+1)}^{(n)} \\ d_1 \xi_{\underline{x}}^{(n)} = \frac{1}{\sqrt{2}} \xi_{(\frac{x_1}{2}, x_2)}^{(n)}, & d_2 \xi_{\underline{x}}^{(n)} = \frac{1}{\sqrt{2}} \xi_{(x_1, \frac{x_2}{2})}^{(n)} \end{cases} \tag{5.4}$$

which is all that we need in the following.

5.2. Wavelets o.n. bases in any dimension

The problem we want to solve here is completely analogous to that considered in section 3. For simplicity we fix here $d = 2$, but the extension to higher dimensions is straightforward. More in details we want to construct a square-integrable function, which we again call $\Psi^{(o)}(\underline{x})$, such that the set

$$\mathcal{F} = \left\{ \Psi_{\underline{j},\underline{k}}^{(o)}(\underline{x}) = D_1^{j_1} D_2^{j_2} T_1^{k_1} T_2^{k_2} \Psi^{(o)}(\underline{x}), \underline{k}, \underline{j} \in \mathbb{Z}^2 \right\} \tag{5.5}$$

is an o.n. basis in $\mathcal{L}^2(\mathbb{R}^2)$.

Before starting with our construction, it is worth noting that as before this set is only apparently the set of dilated and translated of what we still call the *mother wavelet* $\Psi^{(o)}(\underline{x})$. In other words, even if the operators D_j and T_j look like dilation and translation operators, they behave as these operators only in the *new* variables. Hence, in general, we have $\Psi^{(o)}_{\underline{j},\underline{k}}(\underline{x}) \neq 2^{(j_1+j_2)/2}\Psi^{(o)}(2^{j_1}x_1 - k_1, 2^{j_2}x_2 - k_2)$ (but for trivial canonical transformations, of course). Nevertheless, it is not hard to compute the way in which the operators D_j and T_j act on $\Psi^{(o)}(\underline{x})$. The strategy adopted is completely identical to that in (3.2) and produces, making use of (5.4),

$$\Psi^{(o)}_{\underline{j},\underline{k}}(\underline{x}) = 2^{(j_1+j_2)/2}\int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, K(\underline{x};\underline{y})\Psi^{(n)}(2^{\underline{j}}\underline{y} - \underline{k}), \tag{5.6}$$

where $\underline{j},\underline{k} \in \mathbb{Z}^2$ and we have introduced the compact notation $2^{\underline{j}}\underline{y} - \underline{k} = (2^{j_1}x_1 - k_1, 2^{j_2}x_2 - k_2)$.

First of all we want these functions to be mutually o.n. Once again we start considering the following scalar product in $\mathcal{L}^2(\mathbb{R}^2)$:

$$\tilde{\Omega}_{\underline{j},\underline{k}} := \left\langle\Psi^{(o)}_{\underline{j},\underline{k}}, \Psi^{(o)}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \mathrm{d}\underline{x}\, \overline{\Psi^{(o)}_{\underline{j},\underline{k}}(\underline{x})}\Psi^{(o)}(\underline{x})$$

which, using (5.6) and the resolution of the identity for $\xi^{(n)}_{\underline{y}}$, can be written as

$$\tilde{\Omega}_{\underline{j},\underline{k}} = 2^{(j_1+j_2)/2}\int_{\mathbb{R}^2} \mathrm{d}\underline{y}\, \overline{\Psi^{(n)}(2^{\underline{j}}\underline{y} - \underline{k})}\Psi^{(n)}(\underline{y}).$$

Furthermore, if we start with a factorizable $\Psi^{(n)}(\underline{y})$, $\Psi^{(n)}(\underline{y}) = h_1(y_1)h_2(y_2)$, this can be rewritten as

$$\tilde{\Omega}_{\underline{j},\underline{k}} = \left(2^{j_1/2}\int_{\mathbb{R}} \mathrm{d}y_1\, \overline{h_1(2^{j_1}y_1 - k_1)}h_1(y_1)\right)\left(2^{j_2/2}\int_{\mathbb{R}} \mathrm{d}y_2\, \overline{h_2(2^{j_2}y_2 - k_2)}h_2(y_2)\right). \tag{5.7}$$

In analogy with what we have done for Gabor systems, we now assume that the two sets of functions $\mathcal{E}_l = \{2^{j/2}h_l(2^jx - k), j, k \in \mathbb{Z}\}, l = 1, 2$, are both o.n. wavelets bases of $\mathcal{L}^2(\mathbb{R})$. If this is so, then it is clear that $\tilde{\Omega}_{\underline{j},\underline{k}} = \delta_{\underline{j},\underline{0}}\delta_{\underline{k},\underline{0}}$. This, in turn, because of the commutation rules in (5.2), implies that

$$\left\langle\Psi^{(o)}_{\underline{j},\underline{k}}, \Psi^{(o)}_{\underline{n},\underline{m}}\right\rangle_{\mathcal{L}^2(\mathbb{R}^2)} = \delta_{\underline{j},\underline{n}}\delta_{\underline{k},\underline{m}} \tag{5.8}$$

which is what we had to prove first.

The next step is to prove the completeness in $\mathcal{L}^2(\mathbb{R}^2)$ of the set $\mathcal{F}$. This is guaranteed by the assumption that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are complete in $\mathcal{L}^2(\mathbb{R})$, and the proof is left to the reader since it does not differ significantly from that given for Gabor systems. Therefore $\mathcal{F}$ is an o.n. wavelet-like basis in $\mathcal{L}^2(\mathbb{R}^2)$.

As already mentioned the extension of the procedure to $d > 2$ is not particularly difficult and follows the same steps as for the Gabor-like basis.

Example. $h_1(x) = h_2(x) = H(x)$, where $H(x)$ is the Haar mother wavelet.

We consider first the choice (4.3) of the kernel, which corresponds to the canonical transformation arising from the quantum Hall effect. In this case we get

$$\Psi^{(o)}(\underline{x}) = \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}x_1x_2/2}}{2\pi}\left(\int_0^{1/2} \mathrm{e}^{\mathrm{i}x_1y_1}\frac{(\mathrm{e}^{\mathrm{i}(x_2-y_1)} - 1)^2}{x_2 - y_1}\,\mathrm{d}y_1 - \int_{1/2}^{1} \mathrm{e}^{\mathrm{i}x_1y_1}\frac{(\mathrm{e}^{\mathrm{i}(x_2-y_1)} - 1)^2}{x_2 - y_1}\,\mathrm{d}y_1\right)$$

and figure 4 shows the plot of its modulus. In this case, finding the action of the operators D_j and T_j on $\Psi^{(o)}(\underline{x})$ is harder than in section 3, and formula (5.6) should be used.

Much easier is the computation of $\Psi^{(o)}(\underline{x})$ using the other two-dimensional kernel, (4.5). In this case we find

$$\Psi^{(o)}(\underline{x}) = \frac{1}{2\pi x_1(x_1 - x_2)}(\mathrm{e}^{\mathrm{i}x_1/2} - 1)^2(\mathrm{e}^{\mathrm{i}(x_2-x_1)/2} - 1)^2$$

which, as expected, is not factorized. Again, $\Psi^{(o)}_{\underline{j},\underline{k}}(\underline{x})$ is given by (5.6).

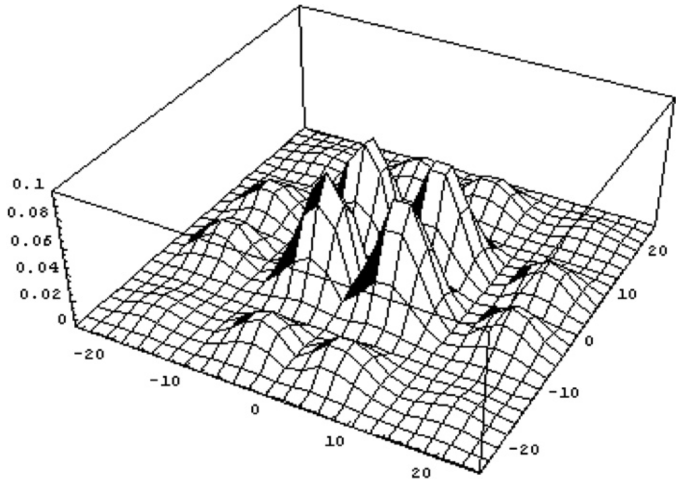

Figure 4. $|\Psi^{(o)}(\underline{x})|$ for $h_1(x) = h_2(x) = H(x)$.

6. Conclusions

In this paper we have constructed an easy technique, based on quantum canonical transformations, which allow us to construct a function $\Psi^{(o)}(\underline{x})$ in $\mathcal{L}^2(\mathbb{R}^d)$ out of d functions in $\mathcal{L}^2(\mathbb{R})$ such that:

- if these functions generate Gabor o.n. bases in $\mathcal{L}^2(\mathbb{R})$, then $\Psi^{(o)}(\underline{x})$ generates a Gabor-like o.n. basis in $\mathcal{L}^2(\mathbb{R}^d)$;
- if these functions generate only Gabor frames in $\mathcal{L}^2(\mathbb{R})$, then $\Psi^{(o)}(\underline{x})$ generates a Gabor-like frame in $\mathcal{L}^2(\mathbb{R}^d)$;
- if these functions generate wavelet o.n. bases in $\mathcal{L}^2(\mathbb{R})$ then $\Psi^{(o)}(\underline{x})$ generates a wavelet-like o.n. basis in $\mathcal{L}^2(\mathbb{R}^d)$.

The reason why we call these sets *Gabor-like* and *wavelet-like* rather than simply *Gabor* and *wavelet* is related to the fact that we are using, as we have explained above, translation, modulation and dilation operators in the *new* variables rather than in the *old* ones. These are still unitary operators but act on $\Psi^{(o)}(\underline{x})$ in a slightly modified way, and therefore they do not produce the standard sets of functions.

The next step of our analysis will be concerned with possible physical applications of our construction, in the attempt of mimicking and extending our old results on the quantum Hall effect.

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